What is the Size of the Hilbert Hotel's Computer?

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Abstract. *The Hilbert's Hotel is a hotel with countably infinitely many rooms. The size of its hypothetical computer was the pretext in order to think about whether it makes sense and what would be* $\log_2(\aleph_0)$*. Thus, at the road of this journey, this little paper demonstrates – surprisingly – that there exist countably infinite sets strictly smaller than* $\mathbb{N}$ *(the natural numbers), with very elementary mathematics, so shockingly stating the inconsistency of the Zermelo-Fraenkel Set Theory with the Axiom of Choice (ZFC).*

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Contents

1. Introduction

This paper proves, utilizing the suitable axioms and rules strictly within ZFC, the inconsistency of the proper ZFC. [7, 16] The proof relies on the construction of a countably infinite set strictly smaller than $\mathbb{N}$, which would be impossible, by the *Axiom of Countable Choice* or *Axiom of Denumerable Choice* (AC_ω) [8], hence this axiom is unfortunately contradictory with ZFC, which implies that the ZFC is inconsistent, regrettably.

Suppose that the size of a computer is, coarse mode, determined by the size (and quantity) of its internal registers and memory (RAM) [14], mainly when these are huge components. Assume yet that a computer, in order to help controlling the administration of a hotel, must be able to cope efficiently online at least with its number of rooms and guests. Presume also if this [finite] number is **n**, then, in order to maximize the speed of the processing, the size of its internal registers and memory cells should be at least about $\lceil \mathbf{\log_2(n)} \rceil$, where the size of the computer would be proportional to $\lceil \mathbf{\log_2(n)} \rceil$. [14]

So, with those same [seemingly sensible] assumptions, what would be the [theoretical] size of a computer for the Hilbert's Hotel, [15] in order to help to manage countably infinitely many rooms and guests ($\aleph_0$)? What about its size being about $\mathbf{\log_2(\aleph_0)}$? Would that question

make sense in ZFC? [1] What about this problem maybe shed light on the question concerning even to consistency of the proper ZFC? [7, 16]

2. Definition of $\log^{-i}$-panfinite sets ($\log^{-i}$-ω), $\log_2(\aleph_0)$ and Γ_{-i}

Definition 2.1: $\log^{-1}$-panfinite set ($\log^{-1}$-ω) and $\log_2(\aleph_0)$. Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be infinite subsets of the natural numbers $\mathbb{N}$ [19], and the real numbers $\mathbb{R}$ [10], respectively, by the *Axiom Schema of Replacement* [1] with the functions below, and let $\boldsymbol{Z}$ be the power set of $\boldsymbol{X}$, by the *Axiom of Power Set* [1]:

$\boldsymbol{X = \{\lfloor \log_2(n) \rfloor : n \in \mathbb{N}^+ \backslash \{1\}\}}$ ($\lfloor log_2 \rfloor : \mathbb{N}^+ \backslash \{1\} \rightarrow \mathbb{N}^+$, where $\lfloor x \rfloor = \max\{m \in \mathbb{Z} \mid m \leq x\}$)

$\boldsymbol{Y = \{1/(\log_2(n) + 1) : n \in \mathbb{N}^+\}}$ ($1/(log_2 + 1) : \mathbb{N}^+ \rightarrow \mathbb{R}$, where $\log_2(n) = y \Leftrightarrow 2^y = n$)

$\boldsymbol{Z}$ **= power set of** $\boldsymbol{X}$

Notice that that set $\boldsymbol{X}$ is very interesting, because even though it is naturally countably infinite, its cardinality or size (n($\boldsymbol{X}$) or $|\boldsymbol{X}|$, that we shall call $\boldsymbol{log_2(\aleph_0)}$ hereafter) is strictly less than the cardinality of $\mathbb{N}$ (where $|\mathbb{N}| = \aleph_0$), by the following theorem:

Theorem 2.1. $\boldsymbol{\log_2(\aleph_0) = |X| < |Y| = |\mathbb{N}| = \aleph_0}$.

Proof. There exists an injective function $\boldsymbol{f : Z \rightarrow Y}$. We can see it defining below the sets $\boldsymbol{X_r}$, $\boldsymbol{Y_r}$ and $\boldsymbol{Z_r}$ and then demonstrating constructively that always $|\boldsymbol{Z_r}| \leq |\boldsymbol{Y_r}|$, for every $\boldsymbol{r}$, and then as $\mathbf{r}$ approaches $\aleph_0$, this shall necessarily lead to $|\boldsymbol{Z}| \leq |\boldsymbol{Y}|$, which implies that $|\boldsymbol{X}| < |\boldsymbol{Y}|$.

Schema of Definition 2.2: Restricted Sets $\boldsymbol{X_r}$, $\boldsymbol{Y_r}$ and $\boldsymbol{Z_r}$. Let the sets $\boldsymbol{X_r}$, $\boldsymbol{Y_r}$ and $\boldsymbol{Z_r}$ be defined as $\boldsymbol{X}$, $\boldsymbol{Y}$ and $\boldsymbol{Z}$ above, but with a set {1, 2, 3, ..., r} replacing $\mathbb{N}^+$, where $r \in \mathbb{N}^+$, or $r = \aleph_0$ (where, in this latter case: $\{1, 2, 3, ..., \aleph_0\} = \{1, 2, 3, ...\} = \mathbb{N}^+$, $X_r = X_{\aleph_0} = X$, $Y_r = Y_{\aleph_0} = Y$ and $Z_r = Z_{\aleph_0} = Z$):

$\boldsymbol{X_r = \{\lfloor \log_2(n) \rfloor : n \in \{2, 3, ..., r\}\}}$

$\boldsymbol{Y_r = \{1/(\log_2(n) + 1) : n \in \{1, 2, 3, ..., r\}\}}$

$\boldsymbol{Z_r}$ **= power set of** $\boldsymbol{X_r}$

So, there exists an injective function $\boldsymbol{f_r : Z_r \rightarrow Y_r}$ for every $\boldsymbol{r}$. We can demonstrate it defining that function as $f_r(\emptyset) = 0$, and for every nonempty subset $\boldsymbol{s} = \{k_1, k_2, \ldots, k_m, \ldots\}$ of $\boldsymbol{Z_r}$, $f_r(\boldsymbol{s}) = 1/(\log_2(1 + 2^{k_1} + 2^{k_2} + \ldots + 2^{k_m} + \ldots) + 1)$. Note that that $\boldsymbol{s}$ can be either a finite or infinite subset of $\boldsymbol{Z_r}$.

Then, we can prove that that f_r is really injective by construction, where for every member $\boldsymbol{p}$ of $\boldsymbol{Z_r}$, there exists one single member $\boldsymbol{y}$ of $\boldsymbol{Y_r}$, that is if $f_r(\boldsymbol{p}) = \boldsymbol{y}$, and $f_r(\boldsymbol{q}) = \boldsymbol{y}$, then $\boldsymbol{p} = \boldsymbol{q}$. This happens because we need double $\mathbf{r}$ in order to generate only one new value to $\lfloor \log_2(\mathbf{r}) \rfloor$, which in its turn will double the sizes of $\boldsymbol{Y_r}$ and of $\boldsymbol{Z_r}$, equalizing exactly their sizes ($|\boldsymbol{Y_r}|$ and $|\boldsymbol{Z_r}|$) when $\boldsymbol{r}$ is a power of 2 (otherwise, if $\boldsymbol{r}$ is *not* a power of 2, then $|\boldsymbol{Y_r}| > |\boldsymbol{Z_r}|$, but these sizes are always *near*: ($|\boldsymbol{Y_r}| - |\boldsymbol{Z_r}|) < 2^r$), since if $k_m \in \boldsymbol{X_r}$, then $\boldsymbol{Y_r}$ and $\boldsymbol{Z_r}$ contain necessarily at least 2^{k_m} elements (or members), which implies, as all $k_m \in \mathbb{N}^+$, that $|\boldsymbol{Y_r}| \geq |\boldsymbol{Z_r}|$ for all $\boldsymbol{r}$ varying from $\mathbf{1}$ up to $\aleph_0$, as shown in the symbolical constructive completed infinite table below:

r	$1/(\log_2(r)+1)$	$\lvert Y_r\rvert$	$\lfloor\log_2(r)\rfloor$	$\lvert Z_r\rvert$	$f : Z_r \to Y_r$
1	1	1	Ø	1	$f(Ø) = 1$
2	0.5	2	1	2	$f(Ø) = 1$
3	0,386…	3	1	2	$f(\{1\}) = 0.5$
4	0.333…	4	2	4	$f(Ø) = 1$
5	0.301…	5	2	4	$f(\{1\}) = 0.5$
6	0.279…	6	2	4	$f(\{2\}) = 0,386…$
7	0.262…	7	2	4	$f(\{1, 2\}) = 0.333…$
8	0.25	8	3	8	$f(Ø) = 1$
9	0.239…	9	3	8	$f(\{1\}) = 0.5$
10	0.231…	10	3	8	$f(\{2\}) = 0,386…$
11	0.224…	11	3	8	$f(\{1, 2\}) = 0.333…$
12	0.218…	12	3	8	$f(3) = 0.301…$
13	0.212…	13	3	8	$f(\{1, 2\}) = 0.279…$
14	0.208…	14	3	8	$f(\{2, 3\}) = 0.262…$
15	0.203…	15	3	8	$f(\{1, 2, 3\}) = 0.25$
16	0.2	16	4	16	$f(Ø) = 1; f(\{1\}) = 0.5; f(\{2\}) = 0.386…; …; f(\{1, 2, 3, 4\}) = 0.2$
…	…	…	…	…	$f(Ø) = 1; f(\{1\}) = 0.5; f(\{2\}) = 0.386…; …; f(\{1, 2, 3, 4, …\}) = …$
2^k	$1/(k + 1)$	2^k	k	2^k	$f(Ø) = 1; f(\{1\}) = 0.5; f(\{2\}) = 0.386…; …; f(\{1, 2, 3, 4, …, k\}) = 1/(k + 1)$
…	…	…	…	…	$f(Ø) = 1; f(\{1\}) = 0.5; f(\{2\}) = 0.386…; …; f(\{1, 2, 3, 4, …\}) = …$
$\aleph_0$	$1/(\log_2(\aleph_0)+1)$	$\aleph_0$	$\log_2(\aleph_0)$	$\aleph_0$	$f(Ø) = 1; f(\{1\}) = 0.5; f(\{2\}) = 0.386…; …; f(\{1, …, \log_2(\aleph_0)\}) = 1/(\log_2(\aleph_0)+1)$

Table 2.1 Symbolical table with the *infinite completed construction* of all $\boldsymbol{Y_r}$ and $\boldsymbol{Z_r}$, varying $\boldsymbol{r}$ from **1** up to $\boldsymbol{\aleph_0}$

Hence, for every finite or infinite subset $\{k_1, k_2, …, k_m, …\}$ of $\boldsymbol{Z}$, there exists a definite and distinct value $1/(\log_2(1 + 2^{k_1} + 2^{k_2} + … + 2^{k_m} + …) + 1)$ of $\boldsymbol{Y}$: So, there is an injective function $\boldsymbol{f : Z \to Y}$, and then $|\boldsymbol{Y}| = |\mathbb{N}| \geq |\boldsymbol{Z}|$, thus we can define $\log_2(\aleph_0) = |\boldsymbol{X}| < |\mathbb{N}| = \aleph_0$, because $|\boldsymbol{X}|$ is strictly less than $|\boldsymbol{Z}|$, since always $|\boldsymbol{w}| < |\mathcal{P}(\boldsymbol{w})|$ (every set is strictly smaller than its power set) for every [finite or infinite] set $\boldsymbol{w}$, by the Cantor's Theorem [13]. □

Verify that the *Cantor's diagonal argument* [13] is not valid here in order to attempt to prove that $|\boldsymbol{Z}| > |\mathbb{N}|$, since $\log_2(\aleph_0) < \aleph_0$, so a supposed anti-diagonal sequence from a countably infinite (supposed exhaustive) $\aleph_0$-enumeration cannot generate another indicator function (or characteristic function) different from all the other ones of this $\aleph_0$-enumeration, since the enumeration is $\aleph_0$-length, but that supposed anti-diagonal is only $\log_2(\aleph_0)$-length, as shown constructively in the symbolical table below, where all the supposed anti-diagonal sequences can be in that $\aleph_0$-enumeration *without* being different from any position of their diagonal sequences (otherwise, then it would lead to a contradiction to the exhaustiveness assumption, and then it would prove that $|\boldsymbol{Z}| > |\mathbb{N}|$, after all, as in that Cantor's argument [invalid here]):

Enumeration/Indicator Function	1	2	3		…	n	…	$\log_2(\aleph_0)$
1	**0**	1	1		…	0	…	1
2	1	**1**	1		…	1	…	0
3	1	1	**0**		…	1	…	0
…	…	…	…		**…**	…	…	…
n	1	1	1		…	**0**	…	1
…	…	…	…		…	…	**…**	…
$\log_2(\aleph_0)$	0	1	0		…	1	…	**1**
$\log_2(\aleph_0)+1$	**1**	0	0		…	0	…	0
$\log_2(\aleph_0)+2$	1	**1**	1		…	1	…	1
$\log_2(\aleph_0)+3$	0	0	**1**		…	0	…	0
…	…	…	…		**…**	…	…	…
$\log_2(\aleph_0)+i$ (Supposed anti-diagonal above)	**1**	**0**	**1**		**…**	**1**	**…**	**0**
…	…	…	…		…	…	**…**	…
$2\log_2(\aleph_0)$	1	0	0		…	1	…	**0**
$2\log_2(\aleph_0)+1$	**0**	0	1		…	0	…	1
$2\log_2(\aleph_0)+2$	1	**1**	0		…	1	…	0
$2\log_2(\aleph_0)+3$	1	1	**1**		…	0	…	1
…	…	…	…		**…**	…	…	…
$2\log_2(\aleph_0)+j$	0	1	1		…	**1**	…	1
…	…	…	…		…	…	**…**	…
$3\log_2(\aleph_0)$	1	0	0		…	0	…	**1**
…	…	…	…		…	…	**…**	…
$\aleph_0$	0	1	0		…	1	…	**0**

Table 2.2 Symbolical table with the *constructive demonstration* that Cantor's diagonal argument is not valid here

In order to better understanding of the infinite construction above, let ***W*** be a set very similar to ***X***, but a finite set instead of an infinite one, for instance, $\boldsymbol{W = \{\lfloor \log_2(n) \rfloor : n \in \{2, 3, 4, 5, 6, 7, 8\}\}}$. What would be $|\boldsymbol{W}|$ here? $|\boldsymbol{W}| = |\{1, 2, 3\}| = 3$.

Notice that $|\boldsymbol{W}| = 3 = \lfloor \log_2(8) \rfloor$, and for every finite or infinite set **{2, 3, …, m}** replacing the **{2, 3, 4, 5, 6, 7, 8}** above, we would have $|\boldsymbol{W}| = \lfloor \log_2(\mathbf{m}) \rfloor$, that is this simple mathematical process allows sensibly to define *integer logarithm of either finite or infinite sets*: Hence, for that $\mathbf{m} = \aleph_0$, we can see clearly that $\log_2(\aleph_0) = |\boldsymbol{W}|$.

So, we shall call that set ***X*** a *log^{-1}-panfinite set* (*log^{-1}-ω*), where its size $|\boldsymbol{X}|$ is the symbol *$log_2(\aleph_0)$*, as defined above, hence $|\boldsymbol{X}| = \log_2(\aleph_0) < \aleph_0$, and $|\boldsymbol{Z}| = |\mathcal{P}(\boldsymbol{X})| = 2^{\log_2(\aleph_0)} = \aleph_0$.

Therefore, four questions loom about that set ***X***, which are readily answered here:

1. "– Is ***X*** really a well-defined set within ZFC?"

– Yes, ***X*** is very well-defined, since its definition results from ZFC, plainly.

2. "– Aprioristically, ***X*** could even be a finite set; so, is ***X*** actually infinite?"

– Yes, it is infinite, since for every number $\lfloor \log_2(r) \rfloor$, there is another one greater than it $\lfloor \log_2(r+r) \rfloor = \lfloor \log_2(r) \rfloor + 1$ (see by the way that we "need" **r** more in the "input" in order to get only 1 more in the "output", which even assist to explain why that set ***X*** "raises" so sluggishly).

3. "– Then, isn't ***X*** in fact a traditional countably infinite set, as $\mathbb{N}$, with cardinality equal to $\aleph_0$ (that is isn't simply $|\boldsymbol{X}| = \aleph_0$)?"

– No, ***X*** cannot be $\aleph_0$-sized, since its cardinality, $\log_2(\aleph_0)$, must be strictly less than $\aleph_0$, as proven within the completed infinite construction shown in the Tab. 2.1 above, unless we conclude otherwise that $|\boldsymbol{Z}| = |\mathcal{P}(\boldsymbol{X})| = 2^{\aleph_0} = \aleph_0$, which would be even very very worse to ZFC. (See within that construction above that **r** more "steps" (numbers) are necessary in order to insert only 1 more member to ***X***, which even helps to clarify why ***X*** "grows" so slowly (logarithmically) on the number **r** of "steps" or table rows in that construction, and hence it cannot "reach" $\aleph_0$; that however can be "attained" here by ***Y***, growing linearly on **r**.)

4. "– In truth, isn't $2^n : \mathbb{N} \rightarrow \mathbb{N}$ an injective [total] function?"

– No, 2^n, neither every increasing exponential function in ***n***, cannot even be just a [total] function from $\mathbb{N}$ to $\mathbb{N}$, since $2^{\aleph_0} > \aleph_0$. On the other hand, every polynomial in ***n*** is so, because $k.\aleph_0^{\,i} \leq \aleph_0$ [18], for every positive finite numbers ***i***, ***k***. (But $2^n : log^{-1}\text{-}\omega \rightarrow \mathbb{N}$ ***is*** an injective [total] function, as *log^{-1}-ω* is defined herein.)

Definition 2.3: Γ_{-1} (First ***Barbosa panfinite number***)**.** Γ_{-1} is simply another symbol (or name, or label) to represent $\log_2(\aleph_0)$, which leads to $\Gamma_{-1} = \log_2(\aleph_0)$, and $2^{\Gamma_{-1}} = \aleph_0$.

Remember that Γ_{-1} is strictly smaller than $\aleph_0$ ($\Gamma_{-1} < \aleph_0$), since $|\boldsymbol{w}| < |\mathcal{P}(\boldsymbol{w})|$, by the Cantor's Theorem [13], although Γ_{-1} is greater than every positive finite integer **n**.

Definition 2.4: Generalization of Def. 2.1: log^{-i}-panfinite sets (log^{-i}-ω) and $\log_2(\Gamma_{-i})$. We can now easily generalize the definitions in the Defs. 2.1 and 2.2, considering *log^0-ω* = $\mathbb{N}^+$, $\Gamma_0 = \aleph_0$, and replacing Γ_{-i+1} by Γ_{-i} over there (where $i \in \mathbb{N}^+$). In more formal terms:

$\boldsymbol{W = \{\lfloor \log_2(n) \rfloor : n \in log^{-i+1}\text{-}\omega \setminus \{1\}\}}$

The crucial question here is again "– What is the cardinality of $\boldsymbol{W}$?"

With similar symbolical completed infinite construction used in the constructive proof of the Theorem 2.1 above, we shall call $\boldsymbol{W}$ a *log^{-i}-panfinite set* (log^{-i}-ω) and the cardinality of $\boldsymbol{W}$ the symbol $log_2(\Gamma_{-i+1})$, where $|\boldsymbol{W}| = \log_2(\Gamma_{-i+1}) < \Gamma_{-i+1}$, and $|\mathcal{P}(\boldsymbol{W})| = 2^{\log_2(\Gamma_{-i+1})} = \Gamma_{-i+1}$.

Definition 2.5 – Generalization of Def. 2.3: Γ_{-i} (*Barbosa panfinite numbers*). Γ_{-i} is simply a defined symbol (or name, or label) in order to represent $\log_2(\Gamma_{-i+1})$, which leads to $\Gamma_{-i} = \log_2(\Gamma_{-i+1})$, $2^{\Gamma_{-i}} = \Gamma_{-i+1}$, and $\Gamma_{-i} < \Gamma_{-i+1}$ (for all $i \in \mathbb{N}$), where $\Gamma_0 = \aleph_0$, leading to $\Gamma_{-1} = \log_2(\Gamma_0)$, $\Gamma_{-2} = \log_2(\Gamma_{-1})$, and so on. (See that all those Γ_{-i} are positive transfinite numbers, that is they are strictly greater than every positive finite integer **n**).

Consequently, initiating with $\log^{-1}$-ω, $\log_2(\aleph_0)$ and Γ_{-1}, we can apply recursively the definitions 2.4 and 2.5 in order to define $\log^{-i}$-panfinite sets ($\log^{-i}$-ω), $\log_2(\Gamma_{-i+1})$ and all the other Barbosa panfinite numbers (Γ_{-i}) for every positive finite integer **i** > 1.

Note that that concept of *Barbosa panfinite numbers* encompasses the *Beth numbers* (infinite cardinal numbers represented by the symbol $\beth_j$, where $\beth_{j+1} = 2^{\beth_j}$, for all $\mathbf{j} \in \mathbb{N}$) [2], since **i** can be non-positive in the Def. 2.5 above, where $\beth_j = \Gamma_j$, for all the integers $\mathbf{j} \geq 0$, entailing that the Beth numbers are just a *proper* subset of the Barbosa panfinite numbers.

Notice also that the countably infinite recursive process above generates countably infinite cardinalities Γ_{-i}, where all ones are strictly greater than every positive finite number **n**. See yet that $\Gamma_0 = \aleph_0 = \beth_0$, hence there is herein a kind of *positive-negative natural symmetry* generalizing from Beth numbers to Barbosa panfinite numbers.

3. What is the size of the Hilbert Hotel's Computer?

With the definitions above, we can already easily answer that question "– What is the size of the Hilbert Hotel's computer?" – It is equal to Γ_{-1}. See the construction of this answer in the symbolical infinite table below (note that an η-sized binary register or RAM cell can store one and only one number from exactly 2^η distinct ones: $0 \ldots 2^\eta - 1$) [14]:

#Rooms	Range of Numbering	Size of Registers/RAM Cells	Size of Computer
2	0…1	1	proportional to ($\propto$) 1
4	0…3	2	$\propto 2$
…	…	…	…
2^n	$0 \ldots 2^n - 1$	n	$\propto n$
…	…	…	…
$\aleph_0 = 2^{\Gamma_{-1}}$	$0 \ldots 2^{\Gamma_{-1}} - 1$	$\log_2(\aleph_0) = \Gamma_{-1}$	Γ_{-1}

Table 3.1 Symbolical table representing the theoretical size of a computer in function of its numbers ranging

In fact, we now can answer innumerable theoretical questions of same kind, such as:

1. "– What is the [theoretical] length of a sequence of symbols that represents the cardinality of $\mathbb{N}$ ($\aleph_0$) in a base **b** [into a numeral system] [3] *strictly greater than* 1?"

– It is equal to $\log_b(\aleph_0) = \Gamma_{-1}$. (Notice that if that base **b** was equal to 1 (unary base), then that length would be equal to $\aleph_0$, instead of Γ_{-1}.)

2. "– How many months should we [theoretically] invest our savings at [positive] fixed rate of interest, [4] in order to get an $\aleph_0$-moneyed account?"

   – We should do it by Γ_{-1} months.

3. "– How many times should you [theoretically] bend an infinitely malleable paper sheet, in order to get an $\aleph_0$-lengthy thread?"

   – You should do it Γ_{-1} times.

4. "– What is the [theoretical] depth (length) of a *perfect binary tree* [5] that has $\aleph_0$ leaves?"

   – It is equal to Γ_{-1}.

5. "– What is the [theoretical] maximal size of an NFA that can be converted into an exponentially larger DFA?" [6]

   – It is equal to Γ_{-1}.

6. "– What is the [theoretical] size of RAM memory pointers [14] into a computer with Γ_{-1}-sized RAM (that is, its primary address space ranging from 0 to Γ_{-1})?"

   – It is equal to $\Gamma_{-2} = \log_2(\Gamma_{-1})$.

7. "– How many terms are there in the infinite sum that is used as a representation of some Zeno's Paradoxes: $1/2 + 1/4 + \ldots + 1/2^n + \ldots = 1$?" [17]

   – If we consider sensibly that all those terms are rational numbers, then 2^n is upper bounded by $\aleph_0$, hence there are Γ_{-1} terms in that sum.

8. "– Can the *Mathematical Induction* be used in order to establish a given statement for ***all*** $\aleph_0$ natural numbers?" [18]

   – No, in general, it cannot; it can do it only for the first Γ_{-j} natural numbers, where that statement is proven for all ones only when $\boldsymbol{j = 0}$, since only that $\Gamma_0 = \aleph_0$. The maximum increasing rate (polynomial, exponential, etc.) of the integer formulas that occur within each particular induction shall determine that particular ***j***. For instance, the inductive proof that $2^n > n^3$ (for $n \geq 10$) is valid only for the first Γ_{-1} natural numbers, not for all $\aleph_0$ ones, as 2^n is not integer for **n** beyond Γ_{-1}, because naturally $2^{\aleph_0} > \aleph_0$, and $2^{\Gamma_{-1}} = \aleph_0$.

So, like G. Cantor, *je le vois, mais je ne le crois pas!* [12]: – There exist many countably infinite sets strictly smaller than $\mathbb{N}$.

Thus, as a preliminary result, the cardinalities in this paper can be strictly ordered by finite and infinite magnitudes, as simply outlined below:

$0, 1, \ldots, n, \ldots, \Gamma_{-i-1}, \Gamma_{-i}, \ldots, \Gamma_{-1}, \Gamma_0 = \aleph_0 = \beth_0, \Gamma_1, \Gamma_2, \ldots, \Gamma_{p-1}, \Gamma_p, \ldots$

I know that this paper proves a result that apparently is a *nonsense* announcement (the existence of infinite countable sets strictly smaller than $\mathbb{N}$), but it is really a great breakthrough in Set Theory, showing one more time that advances in Mathematics can be originated by ideas and demonstrations that challenge the common sense and the tradition, attaining a higher level of understanding and a deeper layer of comprehension.

4. Changes in the Implications of Peano Axioms for the Natural Numbers

With the demonstration in this paper of the existence of those sets $\mathbf{log^{-i}\text{-}\omega}$ (infinite countable sets strictly smaller than $\mathbb{N}$), we must change the implications of the Peano Axioms for the natural numbers, replacing in this field all statements like "every natural number" with "every natural number in [*some set* $\mathbf{log^{-i}\text{-}\omega}$]", where this set $\mathbf{log^{-i}\text{-}\omega}$ can be the proper $\mathbf{log^{0}\text{-}\omega} = \mathbb{N}^{+}$, naturally, where in, *and only in*, this case that statement can continue as "every natural number".

5. Related Work

The main result of this paper unfortunately asserts that the *Axiom of Countable Choice* or *Axiom of Denumerable Choice* (AC_{ω}) [8] (that states that $\aleph_0$ is smaller than every other transfinite cardinal number) is inconsistent with ZFC (so, the *axiom of choice*, a stronger version of that one) [9], which implies that the ZFC is inconsistent, lamentably.

Therefore, I think we need build a new foundational frame to support and unify the Axiomatic Mathematics, either fixing or replacing the ZFC.

6. Freedom & Mathematics

"**– The essence of Mathematics is Freedom.**" (Georg Cantor) [11]

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